

On a new transformation for generalised porous medium equations: from weak solutions to classical

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Abstract

It is well-known that solutions for generalised porous medium equations are, in general, only Hölder continuous. In this note, we propose a new variable substitution for such equations which transforms weak solutions into classical.

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1 Introduction

Let us consider a generalised porous medium equation in the form

$$\partial_t u = a(u)\Delta u + f(u) \text{ in } (0, T) \times \Omega =: Q_T, \quad (1.1a)$$

$$u = u_\Gamma \quad \text{in } (0, T) \times \Gamma, \quad (1.1b)$$

$$u = u_0 \quad \text{in } \Omega \quad (1.1c)$$

for some $T > 0$ and a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, with a smooth boundary Γ . We assume the diffusion coefficient a to be strictly positive but for $a(0) = 0$. Below, we shall give detailed assumptions on a , the reaction term f , the boundary data u_Γ and the initial data u_0 (see *Assumptions 2.1*). If, for example, $a(u) = m|u|^{1-1/m}$ for some $m > 1$, then equation (1.1a) can be obtained from the standard porous medium equation

$$\partial_t s = \Delta(s|s|^{m-1}) + g(s) \text{ in } Q_T, \quad (1.2)$$

where

$$s := u|u|^{1/m-1}, \quad g(s) := 1/m|s|^{1-m}f(s|s|^{m-1}).$$

It is well-known that solutions of (1.2) are, in general, only weak solutions if $s \not\equiv 0$ and is not strictly separated from zero. In particular, $\partial_t s$ and $\Delta(s|s|^{m-1})$ are generally not even uniformly bounded. Still, it is well-understood [1, 3, 4, 7] that, under reasonable assumptions on g , the solution s is at least Hölder continuous in \overline{Q}_T . The same is all the more true for $u = s|s|^{m-1}$. In this note, we show how the information on the Hölder continuity of solution u can be used in order to transform equation (1.1a) into a generalised porous medium equation with a *classical* solution v . Our construction is based on a new variable substitution $v = V(u)$ by means of a smooth and strictly increasing function V which depends only upon T , the structure of Γ and some Hölder exponents and norms of a , f and u . Thus, although u itself is only Hölder continuous, it can be reconstructed from a regular solution v .

This note is organised as follows: in *Section (2)* we state our assumptions and result, which we then prove in *Section (3)*. Finally, in *Section (4)*, we give an explicit transformation for a solution of the classical homogeneous porous medium equation.

2 Problem setting and main result

In this note, we are interested in a transformation of (4.1) which leads to a classical solution. The latter means that the resulting equation should hold in some Hölder space. We thus assume the reader to be

familiar with the standard and anisotropic Hölder spaces and their standard properties. We make the following structural assumptions upon the parameters and a solution of problem (4.1):

Assumptions 2.1.

1. The diffusion coefficient $a : \mathbb{R} \rightarrow \mathbb{R}_0^+$ has the properties
 - (a) $a(0) = 0$, $a > 0$ in $\mathbb{R} \setminus \{0\}$;
 - (b) a is increasing on \mathbb{R}_0^+ and decreasing on \mathbb{R}_0^- ;
 - (c) $a \in C^{(\alpha_a)}(\mathbb{R} \setminus \{0\})$ for some exponent $\alpha_a \in (0, 1)$;
2. The reaction term $f : \mathbb{R} \rightarrow \mathbb{R}$ has the property: $f \in C^{(\alpha_f)}(\mathbb{R} \setminus \{0\})$ for some exponent $\alpha_f \in (0, 1)$;
3. The function $u : \overline{Q_T} \rightarrow \mathbb{R}$ has the properties
 - (a) u is a solution to initial-boundary value problem (1.1) in a weak [6] sense;
 - (b) $u \in C^{(\alpha_u/2, \alpha_u)}(\overline{Q_T})$ for some exponent $\alpha_u \in (0, 1)$;
 - (c) $u_\Gamma \in C^{(1+\alpha/2, 2+\alpha)}(\{u_\Gamma \neq 0\})$, $u_0 \in C^{(2+\alpha)}(\{u_0 \neq 0\})$ for $\alpha := \min\{\alpha_a, \alpha_f\}\alpha_u$, and the compatibility condition of the first order [5, Chapter IV, §5] holds in the following sense:

$$u_\Gamma(0, \cdot) = u_0, \quad a(u_0)\Delta u_0 + f(u_0) = \partial_t u_\Gamma(0, \cdot) \text{ in } \Gamma \cap \{u_0 \neq 0\}.$$

Under Assumptions 2.1, we prove the following theorem:

Theorem 2.2. Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded domain with a smooth boundary Γ and let Assumptions 2.1 hold. There exist:

1. a strictly increasing function $V \in W^{1,\infty} \left[-\|u\|_{C(\overline{Q_T})}, \|u\|_{C(\overline{Q_T})} \right]$ with $V(0) = 0$,
2. a function $\bar{f}_u \in C^{(\alpha/2, \alpha)}(\overline{Q_T})$,

such that $v := V(u)$ is a classical solution of a generalised porous medium equation:

$$\partial_t v = \nabla \cdot (a \circ U(v) \nabla v) + \bar{f}_u \text{ in } C^{(\alpha/2, \alpha)}(\overline{Q_T}). \quad (2.1)$$

Here, U denotes the inverse of V . Function V depends only upon T , the structure of Γ , the Hölder exponents α_a, α_f and α_u and the function

$$\begin{aligned} \psi : \left[-\|u\|_{C(\overline{Q_T})}, \|u\|_{C(\overline{Q_T})} \right] \setminus \{0\} &\rightarrow \mathbb{R}_0^+, \text{ and for all } k \in \left(0, \|u\|_{C(\overline{Q_T})} \right) \\ \psi(k) &:= \max \left\{ \|a\|_{C^{(\alpha_a)}[k, \|u\|_{C(\overline{Q_T})}]} \|f\|_{C^{(\alpha_f)}[k, \|u\|_{C(\overline{Q_T})}]} \|u\|_{C^{(\alpha_u/2, \alpha_u)}(\{u \geq k\})}, \right. \\ &\quad \left. \|u_\Gamma\|_{C^{(1+\alpha/2, 2+\alpha)}(\{u_\Gamma \geq k\})}, \|u_0\|_{C^{(2+\alpha)}(\{u_0 \geq k\})} \right\}, \end{aligned} \quad (2.2)$$

$$\begin{aligned} \psi(-k) &:= \max \left\{ \|a\|_{C^{(\alpha_a)}[-\|u\|_{C(\overline{Q_T})}, -k]} \|f\|_{C^{(\alpha_f)}[-\|u\|_{C(\overline{Q_T})}, -k]} \|u\|_{C^{(\alpha_u/2, \alpha_u)}(\{u \leq -k\})}, \right. \\ &\quad \left. \|u_\Gamma\|_{C^{(1+\alpha/2, 2+\alpha)}(\{u_\Gamma \leq -k\})}, \|u_0\|_{C^{(2+\alpha)}(\{u_0 \leq -k\})} \right\}. \end{aligned} \quad (2.3)$$

3 Proof of Theorem 2.2

Notation 3.1. To shorten the notation, we make the following convention: if a quantity depends only upon such parameters as T , the structure of Γ , the Hölder exponents α_a, α_f and α_u and the function ψ , we say that it depends only upon the parameters of the problem.

We need some preliminary work in order to define the desired transformation V . Let the Assumptions 2.1 hold. Then, it follows that

$$a(u) \in C^{(\alpha_a \alpha_u/2, \alpha_a \alpha_u)}(\{u \neq 0\}), \quad f(u) \in C^{(\alpha_f \alpha_u/2, \alpha_f \alpha_u)}(\{u \neq 0\}).$$

Moreover, since u is continuous on $\{u \neq 0\}$, the sets $\{u \geq k\}$ ($\{u \leq -k\}$) and $\{u > k\}$ ($\{u < -k\}$) are for all $k \in \left(0, \|u\|_{C(\overline{Q_T})} \right)$ compact and relatively (with respect to $\overline{Q_T}$) open, respectively. Thus, each set $\{u \geq k\}$ ($\{u \leq -k\}$) can be covered by a finite number of relatively (with respect to $\overline{Q_T}$) open cylinders

contained in $\{u > k/2\}$ ($\{u < -k/2\}$). In tern, for each such cylinder Q , there exists a number $\delta_k > 0$ such that the cylinder $Q_{\delta_k} := \{x \in Q_T \mid \text{dist}(x, Q) < \delta_k\}$, where $\text{dist}(x, Q) := \max\{|x - y| \mid y \in Q\}$, is contained in $\{u > k/2\}$ ($\{u < -k/2\}$) as well. Equation (1.1a) is non-degenerate on $\{u > k/2\}$ ($\{u < -k/2\}$). Moreover, the boundary trace u_Γ and the initial value u_0 are regular and compatible on these sets due to *Assumptions 2.1 3(d)*. Therefore, we can apply a standard result on the local regularity of linear parabolic equations with Hölder continuous coefficients, Theorem 10.1 from [5, Chapter IV, §10], to the cylinder Q as subcylinder of Q_{δ_k} . Since $\{u \geq k\}$ ($\{u \leq -k\}$) is covered by a finite number of cylinders of this type, the result of that theorem can be interpreted in the following way: there exists a function

$$\varphi : \left[-\|u\|_{C(\overline{Q_T})}, \|u\|_{C(\overline{Q_T})} \right] \rightarrow \mathbb{R}_0^+$$

with the properties

1. $\varphi(0) = 0$, $\varphi > 0$ in $\left[-\|u\|_{C(\overline{Q_T})}, \|u\|_{C(\overline{Q_T})} \right] \setminus \{0\}$;
2. φ is increasing on $\left[-\|u\|_{C(\overline{Q_T})}, 0 \right]$ and decreasing on $\left[0, \|u\|_{C(\overline{Q_T})} \right]$;
3. for $\alpha = \min\{\alpha_a, \alpha_f\}\alpha_u$, it holds for all $k \in \left(0, \|u\|_{C(\overline{Q_T})} \right]$ that

$$\begin{aligned} \|u\|_{C^{(1+\alpha/2, 2+\alpha)}(\{u \geq k\})} &\leq \varphi^{-1}(k), \\ \|u\|_{C^{(1+\alpha/2, 2+\alpha)}(\{u \leq -k\})} &\leq \varphi^{-1}(-k). \end{aligned}$$

4. φ depends only upon the parameters of the problem.

Without loss of generality, we may also assume that

$$\varphi(k) \leq a^2(k) \text{ for all } k \in \left[-\|u\|_{C(\overline{Q_T})}, \|u\|_{C(\overline{Q_T})} \right]. \quad (3.1)$$

Otherwise, we replace φ by $\min\{\varphi, a^2\}$. Using φ , we construct yet another function

$$\Phi : \left[-\|u\|_{C(\overline{Q_T})}, \|u\|_{C(\overline{Q_T})} \right] \rightarrow \mathbb{R}, \quad \Phi(k) := \int_0^k \int_0^x \left(\int_0^y \varphi(z/2) dz \right)^2 dy dx.$$

It is obvious that $\Phi \in W^{3,\infty} \left[-\|u\|_{C(\overline{Q_T})}, \|u\|_{C(\overline{Q_T})} \right]$. Our next step is to study partial derivatives of $\Phi(u)$. Simple application of the chain rule yields

$$\partial_t \Phi(u) = \Phi'(u) \partial_t u = \int_0^u \left(\int_0^x \varphi(y/2) dy \right)^2 dx \partial_t u, \quad (3.2)$$

$$\nabla \Phi(u) = \Phi'(u) \nabla u = \int_0^u \left(\int_0^x \varphi(y/2) dy \right)^2 dx \nabla u, \quad (3.3)$$

$$\begin{aligned} \partial_{x_i x_j} \Phi(u) &= \Phi'(u) \partial_{x_i x_j} u + \Phi''(u) \partial_{x_i} u \partial_{x_j} u \\ &= \int_0^u \left(\int_0^x \varphi(y/2) dy \right)^2 dx \partial_{x_i x_j} u + \left(\int_0^u \varphi(x/2) dx \right)^2 \partial_{x_i} u \partial_{x_j} u \text{ for } i, j \in 1 : d. \end{aligned} \quad (3.4)$$

In order to gain estimates for (3.2)-(3.4) in $C^{(\alpha/2, \alpha)}(\overline{Q_T})$, we recall two properties of Hölder norms. Let C be a closed subset of an open set $X \subset \mathbb{R}^d$ and let $\beta \in (0, 1]$. For all $g_1, h_1, h_2 \in C^{(\beta)}(X)$, $g_2 \in W^{1,\infty}(g_1(X))$, it holds that

$$\|g_2 \circ g_1\|_{C^{(\beta)}(C)} \leq \|g_2\|_{L^\infty(g_1(C))} + \|\nabla g_2\|_{L^\infty(g_1(C))} \|g_1\|_{C^{(\beta)}(C)}, \quad (3.5)$$

$$\|h_2 h_1\|_{C^{(\beta)}(C)} \leq \|h_1\|_{C^{(\beta)}(C)} \|h_2\|_{C^{(\beta)}(C)}. \quad (3.6)$$

Using (3.5)-(3.6) and the properties of φ , we conclude from (3.2)-(3.4) that

$$\|\Phi(u)\|_{C^{(1+\alpha/2, 2+\alpha)}(\overline{Q_T})} \leq C_1 \quad (3.7)$$

for some $C_1 > 0$ which depends only upon the parameters of the problem. Indeed, for all $i, j \in 1 : d$ and $k \in \left(0, 1/2 \|u\|_{C(\overline{Q_T})} \right]$, it holds, for instance, that

$$\|\Phi''(u) \partial_{x_i} u \partial_{x_j} u\|_{C^{(\alpha/2, \alpha)}(\{k \leq u \leq 2k\})}$$

$$\begin{aligned}
&= \left\| \left(\int_0^u \varphi(x/2) dx \right)^2 \partial_{x_i} u \partial_{x_j} u \right\|_{C^{(\alpha/2, \alpha)}(\{k \leq u \leq 2k\})} \\
&\leq \left\| \int_0^u \varphi(x/2) dx \right\|_{C^{(\alpha/2, \alpha)}(\{k \leq u \leq 2k\})}^2 \|\partial_{x_i} u\|_{C^{(\alpha/2, \alpha)}(\{k \leq u \leq 2k\})} \|\partial_{x_j} u\|_{C^{(\alpha/2, \alpha)}(\{k \leq u \leq 2k\})} \\
&\leq \left\| \int_0^u \varphi(x/2) dx \right\|_{C^{(\alpha/2, \alpha)}(\{k \leq u \leq 2k\})}^2 \varphi^{-2}(k) \\
&\leq \left(\|u\|_{C^{(\alpha/2, \alpha)}(\overline{Q_T})} + 2k \right)^2 \\
&\leq 5 \|u\|_{C^{(\alpha/2, \alpha)}(\overline{Q_T})}^2.
\end{aligned}$$

Similarly, for all $k \in \left(0, \frac{1}{2}\|u\|_{C(\overline{Q_T})}\right]$ it holds that

$$\|\Phi''(u) \partial_{x_i} u \partial_{x_j} u\|_{C^{(\alpha/2, \alpha)}(\{-2k \leq u \leq -k\})} \leq 5 \|u\|_{C^{(\alpha/2, \alpha)}(\overline{Q_T})}^2.$$

Consequently, we obtain that

$$\|\Phi''(u) \partial_{x_i} u \partial_{x_j} u\|_{C^{(\alpha/2, \alpha)}(\{u \neq 0\})} \leq 5 \|u\|_{C^{(\alpha/2, \alpha)}(\overline{Q_T})}^2. \quad (3.8)$$

Since $\Phi(u) \equiv 0$ on $\{u = 0\}$, (3.8) yields

$$\|\Phi''(u) \partial_{x_i} u \partial_{x_j} u\|_{C^{(\alpha/2, \alpha)}(\overline{Q_T})} \leq 5 \|u\|_{C^{(\alpha/2, \alpha)}(\overline{Q_T})}^2. \quad (3.9)$$

Treating the remaining three terms on the right-hand sides of (3.2)-(3.4) in the same way, we obtain the estimate (3.7).

Now we are ready to produce the variable transformation V with the desired properties. We define

$$V : \left[-\|u\|_{C(\overline{Q_T})}, \|u\|_{C(\overline{Q_T})}\right] \rightarrow \mathbb{R}, \quad V(k) := \int_0^k (a^{-1}\Phi')(x) dx.$$

Clearly, $V(0) = 0$. Further, due to the properties of φ , particularly (3.1), it holds that

$$\begin{aligned}
|V'(k)| &= |(a^{-1}\Phi')(k)| = a^{-1}(k) \left| \int_0^k \left(\int_0^x \varphi(y/2) dy \right)^2 dx \right| \\
&\leq \frac{1}{3}|k|^3 a^{-1}(k) \varphi^2(k/2) \leq C_2 \text{ for all } k \in \left[-\|u\|_{C(\overline{Q_T})}, \|u\|_{C(\overline{Q_T})}\right]
\end{aligned}$$

for some constant C_2 which depends only upon the parameters of the problem. Hence, the function V is well-defined and belongs to $W^{1, \infty} \left[-\|u\|_{C(\overline{Q_T})}, \|u\|_{C(\overline{Q_T})}\right]$. It is clear also that V is strictly increasing. Let us now check that $\partial_t V(u) \in C^{(\alpha/2, \alpha)}(\overline{Q_T})$. Again, it is sufficient to consider this function on the sets $\{k \leq u \leq 2k\}$ and $\{-2k \leq u \leq -k\}$ for arbitrary $k \in \left(0, \frac{1}{2}\|u\|_{C(\overline{Q_T})}\right]$ and prove that the Hölder norms are bounded by a constant which is independent of k . So let $k \in \left(0, \frac{1}{2}\|u\|_{C(\overline{Q_T})}\right]$. It holds due to the properties of Φ , (3.5)-(3.6) and condition (3.1) that

$$\begin{aligned}
&\|\partial_t V(u)\|_{C^{(\alpha/2, \alpha)}(\{k \leq u \leq 2k\})} \\
&= \|(a^{-1}\Phi')(u) \partial_t u\|_{C^{(\alpha/2, \alpha)}(\{k \leq u \leq 2k\})} \\
&\leq \|a^{-1}(u)\|_{C^{(\alpha/2, \alpha)}(\{k \leq u \leq 2k\})} \|\Phi'(u)\|_{C^{(\alpha/2, \alpha)}(\{k \leq u \leq 2k\})} \|\partial_t u\|_{C^{(\alpha/2, \alpha)}(\{k \leq u \leq 2k\})} \\
&\leq C_1 (a^{-2}(k) \|a(u)\|_{C^{(\alpha/2, \alpha)}(\{k \leq u \leq 2k\})} + a^{-1}(k)) \varphi^{-1}(k) \\
&\leq C_3 a^{-2}(k) \varphi(k) \leq C_4
\end{aligned} \quad (3.10)$$

for some constant $C_4 > 0$, which once again depends only upon the parameters of the problem. Similar estimates hold on the sets $\{-2k \leq u \leq -k\}$.

Let us now go back to (1.1a) and multiply it by $V'(u)$. We obtain after standard calculation that

$$\begin{aligned}
\partial_t V(u) &= \Phi'(u) \Delta u + (a^{-1}\Phi' f)(u) \\
&= \Delta \Phi(u) - \Phi''(u) |\nabla u|^2 + (a^{-1}\Phi' f)(u) \text{ in } Q_T.
\end{aligned} \quad (3.11)$$

We define

$$\bar{f}_u := -\Phi''(u) |\nabla u|^2 + (a^{-1}\Phi'f)(u) \text{ in } Q_T.$$

Equation (3.11) then reads:

$$\partial_t V(u) = \Delta \Phi(u) + \bar{f}_u \text{ in } Q_T. \quad (3.12)$$

We already know from estimates (3.7) and (3.10) that functions $\Delta \Phi$ and $\partial_t V(u)$ are in $C^{(\alpha/2, \alpha)}(\overline{Q_T})$. Therefore, equation (3.9) holds in $C^{(\alpha/2, \alpha)}(\overline{Q_T})$. Finally, since $\Phi'(u) = (aV')(u)$, we can rewrite (3.12) as

$$\partial_t V(u) = \nabla \cdot (a(u) \nabla V(u)) + \bar{f}_u \text{ in } Q_T. \quad (3.13)$$

For the new variable $v := V(u)$, (3.13) takes the form (2.1):

$$\partial_t v = \nabla \cdot (a \circ U(v) \nabla v) + \bar{f}_u \text{ in } Q_T,$$

where U is the inverse of the function V . This finishes the proof of *Theorem 2.2*. □

4 Example

Let us consider for $m > 1$ the homogeneous porous medium equation

$$\partial_t u = mu^{1-1/m} \Delta u \text{ in } [0, T] \times \overline{B_R} =: \overline{Q_T}, \quad (4.1)$$

where $T > 0$ and B_R is the open d -ball of an arbitrary radius $R > 0$ centred at the origin. One of the solutions of (4.1) is the function

$$u(t, x) = B(t + 1, x) \text{ for all } t \in [0, T], \ x \in B_R,$$

where B is the well-known Barenblatt solution [2]

$$B(t, x) = t^{-\frac{md}{d(m-1)+2}} \left(C - \frac{m-1}{2m(d(m-1)+2)} |x|^2 t^{-\frac{2}{d(m-1)+2}} \right)_+^{\frac{m}{m-1}}$$

for arbitrary constant $C > 0$. Let $\alpha \in (0, 1)$ be arbitrary. We set

$$\Phi(k) := \frac{m(1+\alpha)}{2+\alpha} k^{(1-1/m)(2+\alpha)} \text{ for all } k \in \mathbb{R}_0^+.$$

Clearly, $\Phi(u) \in C^{(2+\alpha)}(\overline{Q_T})$. Following the construction from the proof of *Theorem 2.2*, we introduce the variable transformation

$$v := V(u) := \int_0^u (1/m u^{1/m-1} \Phi')(x) dx = u^{(1-1/m)(1+\alpha)}$$

and the new reaction term

$$\bar{f}_u := -\Phi''(u) |\nabla u|^2 = -(1+\alpha)(m-1)((1-1/m)(2+\alpha)-1)u^{(1-1/m)(2+\alpha)-2} |\nabla u|^2.$$

Again, it is clear that $V(u) \in C^{(1+\alpha)}(\overline{Q_T})$. Thus, we obtain that v is a classical solution of a porous medium equation:

$$\partial_t v = m \nabla \cdot \left(v^{\frac{1}{1+\alpha}} \nabla v \right) + \bar{f}_u \text{ in } C^{(\alpha)}(\overline{Q_T}).$$

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